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Realization of central charges in theories with generalized noncommutative supersymmetry

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Abstract: In a recent paper [14] we have shown that the noncommutative SUSY algebra earlierly proposed in [13] has a natural interpretation and generalization in terms of a U(1) super gauge symmetry, and hence, it must be naturally realized by supersymmetric gauge theories on superspace. Here we further elaborate on this idea by considering a specific example of such models — the U(1) gauged supersymmetric sigma model with a superpotential term in $D = 2$, $\mathcal{N} = 1$ superspace — and investigate the conditions under which the classical dynamics allows for the realization of a central charge in its associated generalized superalgebra. This realization accounts for an elegant interpretation of the central charge as a U(1) charge for the elementary excitations as well as for the solitonic (BPS) solutions of the model under consideration.

Keywords: Sigma Models, Supersymmetric Effective Theories, Superspaces, Non-Commutative Geometry.

Contents

1. Introduction

One of the key ingredients in noncommutative field theories (NCFT's) is their possibility for performing field redefinitions, named as the Seiberg-Witten (SW) maps, enabling one to replace them by equivalent commutative field theories in the presence of some constant B-field background [1, 2]. This seemingly peculiar feature of NCFT's has a simple analogue in the context of ordinary quantum mechanics applied to a charged particle in a constant magnetic field. It is known that the canonical momenta of such a particle behave as non-commuting variables while its coordinates are still commuting. It is possible, however, to convert this problem, via a proper phase space transformation, to the quantum mechanical problem of a particle on a noncommutative space without an external magnetic field [3]–[5].

The above philosophy underlying the SW maps has a natural extension to the case of supersymmetric NCFT's[1] (for reviews on supersymmetry see [18, 19]). Supersymmetric NCFT models have been extensively studied in the literature from several perspectives (see for example [2] and references therein). These are usually considered as the Moyal deforma-

[1] Originally, SW maps were found for the case of a pure noncommutative gauge theory [1]. It was later shown that it can be applied to other NCFT's including a class of noncommutative supersymmetric field theories [8].

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tion of the ordinary SUSY field theories on commutative spaces. Recently, the possibility of deforming the SUSY algebra underlying such theories was investigated in a number of papers [7]–[17]. Along this line of research, it was pointed out in [13] that the noncommutativity of derivatives in a possible description of NCFT's [6], in the case of supersymmetric theories, is manifested in the non-(anti)commutativity of the (super)translation generators, thus leading to a deformation of the superalgebra that ordinarily underlies these theories. This may apparently seem in contradiction to the string theoretic result of [7] by which no such a deformation of the superalgebra on the worldvolume theory of a D-brane in a B-field background is evidenced by the algebra of the string worldsheet supercharges. However, given this result to be correct in a NCFT, by the above interpretation of the SW maps, it is natural to expect the deformation of the algebra somehow to be realized in the commutative description of the same theory. The precise form of this realization has already been given in [14]. The noncommutativity effects in this latter framework are then encoded in the presence of a background commutative gauge field $A_\mu(x)$ with a constant field strength $F_{\mu\nu}$ (related to the noncommutativity parameters) which plays the same role as a $B_{\mu\nu}$ background in the counterpart commutative theory.

One may ask what happens to the deformation of the SUSY algebra if the background $F_{\mu\nu}(x)$ is *not* taken to be constant. The situation is even more interesting if $A_\mu(x)$ belongs to a multiplet of fields defined through a background gauge superfield $\mathcal{A}_\alpha(x,\theta)$. The corresponding U(1) gauge symmetry is then replaced by a local symmetry on superspace [18]. This possibility was investigated in some detail in ref. [14] for the case of $D = 2$, $\mathcal{N} = 1$ theories in euclidean space and it was shown that it can lead to several extensions of the ordinary super E^2 algebra including its central extension (CE-SUSY) and noncommutative deformation (NC-SUSY). The additional generators that naturally appear in the general form of the "extended" (deformed) algebra find interpretations in terms of the local U(1) generators.

It was shown in [14] that such a "generalized superalgebra" is realized in a gauge superfield theory (GSFT) with a field content consisting of a gauge and a scalar superfield, denoted as $\mathcal{A}_\alpha$ and S, respectively. Such a theory possesses two types of symmetries: a global supersymmetry and a local superspace gauge symmetry. The "generalized" supertranslation symmetries of this theory are then realized as a result of "knitting" together these two types of symmetries in a very particular way. There are two different interpretations of the resulting generalized superalgebra. On the one hand, by viewing $\mathcal{A}_\alpha$ as a *fixed* background while varying S only under the generalized supertranslations, we obtain a generalization of the ordinary superalgebra which is closed *only* for particular configurations of $\mathcal{A}_\alpha$. Several such configurations can be found which give rise to alternative generalizations of the ordinary superalgebra. On the other hand, as transformations changing *both* the gauge and matter (scalar) superfields $(\mathcal{A}_\alpha, S)$, the generalized supertranslations are found to form a closed algebra together with the super gauge transformations. In this interpretation, $\mathcal{A}_\alpha$ is no longer treated as a fixed background but it must be viewed as a dynamical variable on equal footing to S.

Working within this latter framework, one must seek a supersymmetric model with a U(1) gauge symmetry as the GSFT model that realizes the above generalized NC-SUSY.

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We work throughout this paper with the simplest of such models consisting of a U(1) gauged supersymmetric σ-model which also involves an additional superpotential term. Our superpotentials generally also depend on $\mathcal{A}_\alpha$ trough its "superfield strength" T, in addition to the usual $(S, \bar{S})$. An interesting question concerning such a model is then to characterize the conditions under which it allows for a classical configuration of $\mathcal{A}_\alpha$ corresponding to a given generalization of the super E^2 algebra. This is important since, for interpreting the generators of the algebra as the *conserved* charges of the theory, the field configurations must be on the equations of motion (EOM).

In particular, we know that a constant configuration of the superfield strength T corresponds to central extension of SUSY [14]. On the other hand, from the ordinary considerations, one expects to realize the central charge as a topological charge in the algebra of supercharges on the BPS solutions of a supersymmetric field theory. This can be explicitly seen, for example, in the case of a Wess-Zumino model in 2D with any well behaved superpotential term [20]. It is therefore interesting to embed the Wess-Zumino model in a GSFT model governing $(S, \mathcal{A}_\alpha)$ and ask for the conditions under which it allows for an arbitrary constant T configuration. This involves finding the change of variables that relates the field contents of the theory in the two formulations. The Wess-Zumino model then emerges as the effective theory of the gauge invariant variable $R \equiv (S\bar{S})^{1/2}$ in which all other field variables are eliminated. The similar procedure may also be carried out for the case of a constant $F \equiv \mathcal{D}_-\mathcal{D}_+T$, which corresponds to the case of NC-SUSY algebra [13]. This has more subtleties than the above case because of the technical difficulties due to a non-constant T configuration. In this paper we are mainly concerned with the case of a constant T as is suitable for realizing the CE-SUSY. The study of the general case, including the one with a constant F suitable for NC-SUSY, is postponed to another paper [15].

This paper is organized as follows: in section 2 we give a brief review of the generalized NC-SUSY as was introduced in the previous work. We then discuss in section 3 some features of the classical dynamics of a U(1) GSFT including a class of the solutions to a pure free GSFT that correspond to some specific classes of the closed super Lie algebras we discussed in the above. In section 4 we add a matter content to the pure GSFT and consider interactions between the gauge and matter multiplets introducing gauge supercovariant derivatives and a gauge invariant superpotential term. We then look for a superpotential that allows for the classical constant T field configurations. We find that all such configurations, except for the BPS ones, after suitable change of variables, are described in terms of an effective Wess-Zumino model with a simple linear superpotential term which hence can be solved for general such configurations. The study of the BPS constant T field configurations is postponed to section 5. We first obtain the effective field theory that governs the gauge invariant degrees of freedom in the GSFT, which is found to be a supersymmetric non-linear σ-model with a specific target space metric and superpotential term. We then write the general BPS equations of this model and demand for the conditions under which it allows for constant T configurations. We conclude the paper with a summary in section 6. The BPS solutions of a generic supersymmetric non-linear σ-model in 2D are discussed in some details in the appendix.

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2. The generalized noncommutative superalgebra

In this section we briefly review the main ideas of [14] for constructing a generalized SUSY algebra using a local gauge symmetry on superspace. In $D = 2$, $\mathcal{N} = 1$ superspace, a gauge supermultiplet is encoded into a grassmann *odd* spinor superfield, $\mathcal{B}_\alpha(x,\theta)$ $(\alpha = \pm)$, with a proper gauge transformation as shall follow. Throughout this paper we adapt to the conventions of [13, 14] for the spinors in a euclidean space and for all our other notations. So, in particular, the spinorial indices are lowered with $\varepsilon_{\alpha\beta}$ and raised with $\varepsilon^{\alpha\beta}$ and the two components of a spinor are related by conjugation as

$$(\mathcal{B}_+)^* = -\mathcal{B}_- \,. \tag{2.1}$$

The field content of $\mathcal{B}_\alpha$ consists of a (complex) scalar, two spinors and a vector field, as can be read easily from its expansion in powers of $\theta^\pm$.

Let us denote the ordinary (undeformed) SUSY generators by $\mathcal{Q}_\alpha$,

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\theta^\pm\partial_\pm \,, \tag{2.2}$$

with $\partial_\pm \equiv \partial/\partial x^\pm$ and $x^\pm \equiv \frac{1}{2}(x \pm iy)$. These operators act from the left on a general complex superfield like $S(x,\theta)$. Then we define the deformed generators Q_α by introducing $\mathcal{B}_\alpha$ as some *superspace gauge connection* as follows

$$Q_\alpha = \mathcal{Q}_\alpha - ie\mathcal{B}_\alpha \,, \tag{2.3}$$

where the dimensionless quantity e denotes the "charge" of the superfield under operation of Q_α. The ordinary algebra of $\mathcal{Q}_\pm$ together with the grassmann property $\{\mathcal{B}_\alpha, \mathcal{B}_\beta\} = 0$ then implies that

$$\begin{aligned} \{Q_+, Q_-\} &= -ie(\mathcal{Q}_+\mathcal{B}_- + \mathcal{Q}_-\mathcal{B}_+) \,, \\ Q_\pm^2 &= i(\partial_\pm - e\mathcal{Q}_\pm\mathcal{B}_\pm) \,. \end{aligned} \tag{2.4}$$

where on the r.h.s., $\mathcal{Q}_\pm$ act on $\mathcal{B}_\pm$ as differential operators on superfields. The expression on the r.h.s. of the anticommutator of Q_+, Q_- is evidently invariant under the following (super) gauge transformations

$$\mathcal{B}_\alpha \to \mathcal{B}_\alpha + \mathcal{Q}_\alpha\Lambda \,, \tag{2.5}$$

where $\Lambda(x,\theta)$ is an arbitrary (parity even) real scalar superfield. This is a gauge transformation in the sense that its parameter Λ is a function of the superspace coordinates. The gauge invariant quantity on the r.h.s. of the first eq. (2.4) is called the *superfield strength* of $\mathcal{B}_\alpha$ which in this paper is denoted by $\hat{T}$. Also the second eq. (2.4), compared to its counterpart in the ordinary superalgebra, hints on defining the generalized momenta as follows

$$P_\pm \equiv -i(\partial_\pm - e\mathcal{Q}_\pm\mathcal{B}_\pm) \,. \tag{2.6}$$

In the following formulae for simplicity we set $e = +1$. We note that, while the above representations of P, Q clearly depend on the gauge in which $\mathcal{B}_\pm$ is written, their algebra is gauge invariant. For example, we see that

$$\begin{aligned}
\{Q_+, Q_-\} &= \hat{T} \equiv -i(\mathcal{Q}_+\mathcal{B}_- + \mathcal{Q}_-\mathcal{B}_+)\,, \\
[P_+, P_-] &= \hat{F} \equiv \mathcal{Q}_-\mathcal{Q}_+\hat{T}\,, \\
[P_+, Q_-] &= \hat{O}_+ \equiv \mathcal{Q}_+\hat{T}\,, \\
[P_-, Q_+] &= -\hat{O}_- \equiv \mathcal{Q}_-\hat{T}\,.
\end{aligned} \tag{2.7}$$

This is a gauge invariant algebra due to the gauge invariance of the combinations of $\mathcal{B}_\pm$ on its r.h.s. Looking on $\hat{T}$, $\hat{F}$, $\hat{O}_\pm$ as operators on the same footing as $P_\pm$, $Q_\pm$, however, we find that their algebra is not in general closed in the sense of a super Lie algebra, provided we treat $\mathcal{B}_\alpha$ as some *fixed* background. Nevertheless, it is closes for very special configurations of $\mathcal{B}_\alpha$ (or more precisely for its superfield strength $\hat{T}$). Two examples of such algebras are the centrally extended supersymmetry (CE-SUSY) and the non-commutative supersymmetry (NC-SUSY) algebras in 2D as alternative generalizations to the ordinary super E^2 algebra. These two cases are characterized by

$$\begin{array}{llll}
\text{CE-SUSY}: & \hat{T} = c\,, & \hat{O}_\pm = 0\,, & \hat{F} = 0\,, \\
\text{NC-SUSY}: & \hat{T} = 2\omega_0\theta^+\theta^-\,, & \hat{O}_\pm = 2\omega_0\theta^\mp\,, & \hat{F} = 2\omega_0\,.
\end{array} \tag{2.8}$$

where c denotes the central charge and ω_0 is the inverse of the non-commutativity parameter. There are a lot of other configurations of $\mathcal{B}_\alpha$ for which the algebra also closes, among which, the exponential configurations of ref. [14] provide alternative examples. Another example is provided as a limit of one of these exponential configurations in the next section.

Finally, we would like to emphasize on the role of $\hat{T}, \hat{O}_\pm, \hat{F}$ as the generators of the gauge transformations in the U(1) GSFT that governs the coupled dynamics of $(S, \mathcal{B}_\alpha)$. In this interpretation, the variations of *both* $(S, \mathcal{B}_\alpha)$ under the "generalized" (super) translations and gauge transformations form a closed algebra without any restrictions.

3. The dynamics of GSFT

In this section we first study some generic features of the classical dynamics of a U(1) GSFT and then specialize to the case of a pure GSFT and, in particular, determine all the classical solutions of a pure free GSFT that lead to closed super Lie algebras of the type described in the previous section. The field content of our models consists of a complex scalar superfield S and a gauge superfield $\mathcal{A}_\alpha$. Both $(S,\, \mathcal{A}_\alpha)$ transform in the same way under the (ordinary) SUSY transformations generated by $\mathcal{Q}_\alpha$,

$$\delta_{\mathcal{Q}} S = (\bar{\epsilon}\mathcal{Q})S\,, \qquad \delta_{\mathcal{Q}}\mathcal{A}_\alpha = (\bar{\epsilon}\mathcal{Q})\mathcal{A}_\alpha\,, \tag{3.1}$$

where $\bar{\epsilon}\mathcal{Q} \equiv \epsilon^\alpha\mathcal{Q}_\alpha$. A supersymmetric superspace lagrangian density for this theory, $\mathcal{L}[S, \mathcal{A}_\alpha]$, is generically written as a function of $(S, \mathcal{A}_\alpha)$ and their supercovariant derivatives $\mathcal{D}_\alpha$ defined as follows

$$\mathcal{D}_\pm \equiv \frac{\partial}{\partial\theta^\pm} - i\theta^\pm\partial_\pm\,. \tag{3.2}$$

Owing to the relations

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = 0\,, \tag{3.3}$$

together with the Leibnitz property of $\mathcal{Q}_\alpha$, we then conclude that

$$\delta_{\mathcal{Q}} \int d^2x d^2\theta\ \mathcal{L} = \int d^2x d^2\theta\,(\bar\epsilon\mathcal{Q})\mathcal{L} = \int d^2x d^2\theta \left(\frac{\partial}{\partial x^a}(\cdots)^a + \frac{\partial}{\partial\theta^\alpha}(\cdots)^\alpha\right) = 0\,. \tag{3.4}$$

Further, for having a U(1) gauge symmetry, $\mathcal{L}$ must be invariant under the so called "super gauge transformations" defined as

$$\delta_\Lambda \mathcal{A}_\alpha = \mathcal{D}_\alpha\Lambda\,, \qquad \delta_\Lambda S = i\Lambda S\,. \tag{3.5}$$

This amounts to replacing all the ordinary supercovariant derivatives in $\mathcal{L}$ with the "gauge supercovariant" derivatives defined as follows

$$D_\alpha \equiv \mathcal{D}_\alpha - ie\mathcal{A}_\alpha\,. \tag{3.6}$$

Here e denotes the U(1) charge of the superfield under operation of D_α (say, $+1, -1, 0$ for $S, \bar S, T$ respectively). So, for example, the kinetic term of $S, \bar S$ in terms of these derivatives become

$$\mathcal{L}_S = \frac{1}{2} D_\alpha S D^\alpha \bar S\,. \tag{3.7}$$

For writing a kinetic term for $\mathcal{A}_\alpha$ we need to introduce the gauge invariant strength of this gauge superfield. Owing to the gauge transformation of $\mathcal{A}_\alpha$,

$$\mathcal{A}_\alpha \to \mathcal{A}_\alpha + \mathcal{D}_\alpha\Lambda\,, \tag{3.8}$$

its superfield strength is defined as follows

$$T \equiv -i(\mathcal{D}_+\mathcal{A}_- + \mathcal{D}_-\mathcal{A}_+)\,. \tag{3.9}$$

The analogous quantities of $\hat O_\pm, \hat F$ (eqs. (2.7)) are also defined

$$\begin{aligned} O_\pm &\equiv \pm\mathcal{D}_\pm T\,, \\ F &\equiv \mathcal{D}_-\mathcal{D}_+ T\,. \end{aligned} \tag{3.10}$$

A proper quadratic lagrangian for $\mathcal{A}_\alpha$ then takes the following form

$$\mathcal{L}_{\mathcal{A}} = \frac{1}{2a}\mathcal{D}_+ T \mathcal{D}_- T\,. \tag{3.11}$$

This leads to a gauge invariant rotationally symmetric and supersymmetric action for $\mathcal{A}_\alpha$ whose EOM for the component fields are at most of second order. Here a is a parameter for setting the scale of T (hence fixing the value of e) which must have the length dimension -2 in order to set the correct dimension -1 for $\mathcal{L}_{\mathcal{A}}$.[2] For simplicity in the following sections

[2]For the action in the x-space to be dimensionless, the superspace action must have dimension $+2$ which, owing to the dimension $+3$ of the measure $d^2xd^2\theta$, implies that a superspace lagrangian density must have the dimension -1. Moreover, we have $[S] = 0$, $[\mathcal{A}_\alpha] = -\frac{1}{2}$.

we adapt units in which $a = 1$. We note, in particular, that when only the scalar or vector components of $\mathcal{B}_\alpha$ survive, the above term give rise to the correct lagrangians of the surviving components,

$$\begin{aligned} \mathcal{A}_\pm(x,\theta) &= \theta^\mp \sigma_\pm(x) \qquad \Rightarrow \qquad \mathcal{L}_\sigma = \frac{1}{2}\partial_+\sigma\,\partial_-\sigma\,, \\ \mathcal{A}_\pm(x,\theta) &= \theta^\pm A_\pm(x) \qquad \Rightarrow \qquad \mathcal{L}_A = -\frac{1}{2}F^2\,, \end{aligned} \tag{3.12}$$

where $\sigma \equiv -i(\sigma_+ + \sigma_-)$ and $F \equiv \partial_+ A_- - \partial_- A_+$.

We recall that the gauge superfields $\mathcal{A}_\alpha, \mathcal{B}_\alpha$, appearing in the GSFT and in the representation of the superalgebra, are generally different due to their different gauge transformations (eqs. (3.8), (2.5)). These transformations imply that the two superfields must be related by

$$\mathcal{B}_+ = C\mathcal{A}_+\,, \qquad \mathcal{B}_- = \bar{C}\mathcal{A}_-\,, \tag{3.13}$$

where the operators $C, \bar{C}$ are defined through their action on a general superfield $\Lambda = \Lambda^{(1)} + \Lambda^{(2)} + \Lambda^{(3)} + \Lambda^{(4)}$ as follows[3]

$$\begin{aligned} C\Lambda &\equiv \Lambda^{(1)} - \Lambda^{(2)} + \Lambda^{(3)} - \Lambda^{(4)}\,, \\ \bar{C}\Lambda &\equiv \Lambda^{(1)} + \Lambda^{(2)} - \Lambda^{(3)} - \Lambda^{(4)}\,. \end{aligned} \tag{3.14}$$

The corresponding strengths of $\mathcal{A}_\alpha$, $\mathcal{B}_\alpha$, namely T, $\hat{T}$ as defined by eqs. (3.9), (2.7), are then related by

$$\hat{T} = C\bar{C}T \qquad \Leftrightarrow \qquad T = C\bar{C}\hat{T}\,. \tag{3.15}$$

The superfield T obeys the correct (ordinary) SUSY transformation rule similar to that of $\mathcal{A}_\alpha$ and S but the SUSY transformation of $\hat{T}$ is slightly different. Indeed, using the above relations one can see that

$$\delta_Q T = (\bar{\epsilon}\mathcal{Q})T \qquad \Leftrightarrow \qquad \delta_Q \hat{T} = -(\bar{\epsilon}\mathcal{Q})\hat{T}\,. \tag{3.16}$$

This shows that (contrary to T) $\hat{T}$ is not a suitable quantity for building a supersymmetric action because of the extra minus sign in its transformation rule.

3.1 The pure GSFT

Let us initially consider the case of a pure GSFT involving only $\mathcal{A}_\alpha$. An appropriate interacting lagrangian for this theory has the following form

$$\mathcal{L}_\mathcal{A} = \frac{1}{2}\mathcal{D}_+ T \mathcal{D}_- T + w(T)\,, \tag{3.17}$$

where $w(T)$ denotes some self-interaction superpotential term. The variation of $\mathcal{L}_\mathcal{A}$ with respect to $\mathcal{A}_\pm$ then becomes

$$\delta\mathcal{L}_\mathcal{A} \cong i\left[\mathcal{D}_+\left(F + \frac{\partial w}{\partial T}\right)\delta\mathcal{A}_- + \mathcal{D}_-\left(F + \frac{\partial w}{\partial T}\right)\delta\mathcal{A}_+\right], \tag{3.18}$$

[3]In agreement with the notation of [14], this decomposition means expanding Λ in powers of $\theta^\pm$ in which $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)}$ stand for the terms proportional to $1, \theta^+, \theta^-, \theta^+\theta^-$, respectively.

where the $\cong$ sign denotes equivalence up to a total derivative and F is defined as in eq. (3.10). The EOM of $\mathcal{A}_\pm$ thus is written as

$$\mathcal{D}_\pm\left(F+\frac{\partial w}{\partial T}\right)=0 \qquad \Rightarrow \qquad F+\frac{\partial w}{\partial T}=\text{const.} \tag{3.19}$$

In particular, in the case of a "free" gauge multiplet with $w(T) = 0$, the EOM of $\mathcal{A}_\alpha$ becomes $F = F_0$ (= constant). The two special configurations of $\mathcal{A}_\alpha$ corresponding to the NC-SUSY and CE-SUSY algebras are therefore solutions to the "free field" EOM of $\mathcal{A}_\alpha$. This is because in both cases F is a constant (see eqs. (2.8)). This raises the question that if there are other solutions to the free field EOM of $\mathcal{A}_\alpha$ on which the generalized SUSY algebra closes. The answer to this question follows from a precise analysis of eq. (3.19) with $w(T) = 0$ which can be written as

$$\mathcal{D}_-\mathcal{D}_+T=F_0\,. \tag{3.20}$$

After solving this equation for $T(x,\theta)$, the corresponding solution for $\mathcal{A}_\alpha(x,\theta)$ is found by inverting eq. (3.9) modulo gauge transformations. The general solution to the eq. (3.20) is easily obtained, if we write it in terms of the component fields of T which are defined through

$$T=\sigma(x)+\theta^+\chi_+(x)+\theta^-\chi_-(x)+\theta^+\theta^-F(x)\,. \tag{3.21}$$

The EOM of $\sigma, \chi_\pm, F$ then become

$$\partial_+\partial_-\sigma=0\,, \qquad \partial_\pm\chi_\mp=0\,, \qquad F=F_0\,. \tag{3.22}$$

These are easily solved and give the general solution for T. For the following purpose, we need to work with the dual variables $\hat{T}, \hat{O}_\pm, \hat{F}$. The general solution in terms of $\hat{T}$ becomes

$$\hat{T}(x,\theta)=f_+(x^+)+f_-(x^-)-\theta^+\chi_+(x^+)-\theta^-\chi_-(x^-)+F_0\theta^+\theta^-\,, \tag{3.23}$$

where for the reality of $\hat{T}$ we need $(f_+)^* = f_-$ and $(\chi_+)^* = -\chi_-$. The corresponding solution for $\hat{O}_\pm$ then becomes

$$\begin{aligned}\hat{O}_+(x,\theta) &= -\chi_+(x^+)+i\theta^+\partial_+f_+(x^+)+\theta^-F_0\,,\\ \hat{O}_-(x,\theta) &= \chi_-(x^-)-i\theta^-\partial_-f_-(x^-)+\theta^+F_0\,.\end{aligned} \tag{3.24}$$

We observe that, apart from a gauge transformation, all the arbitrariness in the general solution to the free field EOM of $\mathcal{A}_\alpha$ comes through the choices of the holomorphic functions $f_+(x^+)$ and $\chi_+(x^+)$ and of the constant F_0.

3.2 Closure of the algebra in a pure free GSFT

We now note that the combinations on the r.h.s. of the generalized superalgebra with the generators $(P_\pm, Q_\pm, \hat{O}_\pm, , \hat{T}, \hat{F})$ are proportional to either of the operators $\hat{T}, \hat{F}, \hat{O}_\pm$ or their derivatives $\partial_\pm\hat{T}, \partial_\pm\hat{F}, \partial_\pm\hat{O}_\pm$ [14]. For being a closed super Lie algebra, the latter must be written as linear combinations of the former. Using the above solution, it is then easy to see that this requirement is satisfied if and only if $\chi_\pm$ are constants and $f_\pm$ have the following forms

$$f_\pm(x^\pm) = \kappa_\pm x^\pm + c_\pm\,, \tag{3.25}$$

with $\kappa_+ = \kappa_-^*$, $c_+ = c_-^*$ being arbitrary constants. Taking $\chi_\pm = 0$, the corresponding solutions for $\hat{T}, \hat{O}_\pm, \hat{F}$ become

$$\begin{aligned} \hat{T}(x,\theta) &= c + \kappa_+ x^+ + \kappa_- x^- + F_0\theta^+\theta^-\,, \\ \hat{O}_+(x,\theta) &= i\kappa_+\theta^+ + F_0\theta^-\,, \\ \hat{O}_-(x,\theta) &= F_0\theta^+ - i\kappa_-\theta^-\,, \\ \hat{F}(x,\theta) &= F_0\,. \end{aligned} \tag{3.26}$$

where $c \equiv c_+ + c_-$ is an arbitrary real parameter. The corresponding solution for $\mathcal{A}_\alpha$ up to a gauge transformation is written as

$$\mathcal{A}_\pm = \pm\frac{1}{2}F_0 x^\mp\theta^\pm + \left(\kappa_\pm x^\mp + \frac{c}{2}\right)\theta^\mp\,. \tag{3.27}$$

This is the most general solution of the free field EOM of $\mathcal{A}_\alpha$ for which the algebra closes.[4] This general solution is evidently a superposition of the three special solutions defined by

$$\begin{array}{llll} \text{CE-SUSY}: & \kappa_\pm = 0\,, & F_0 = 0\,, & c \neq 0\,, \\ \text{NC-SUSY}: & \kappa_\pm = 0\,, & F_0 \neq 0\,, & c = 0\,, \\ \text{LX-SUSY}: & \kappa_\pm \neq 0\,, & F_0 = 0\,, & c = 0\,. \end{array} \tag{3.28}$$

The first two cases obviously correspond to SO(2) symmetric configurations while in the last case the SO(2) symmetry is broken by a preferred direction defined by $\kappa_\pm$. This means, in particular, that in the first two cases the algebra closes also with inclusion of the rotation generator J, while in the third case it does not. This can be checked also explicitly. For example,

$$\begin{aligned} [J,\hat{T}] &= \kappa_+ x^+ - \kappa_- x^-\,, \\ [J,\hat{O}_+] &= \frac{1}{2}(-F_0\theta^- + i\kappa_+\theta^+)\,, \\ [J,\hat{O}_-] &= \frac{1}{2}(F_0\theta^+ + i\kappa_-\theta^-)\,, \\ [J,\hat{F}] &= 0\,, \end{aligned} \tag{3.29}$$

from which we see that the commutator of J with other generators is not closed unless for $\kappa_\pm = 0$.[5]

[4] Indeed, with a non-zero constant $\chi_\pm$, a term of the form $-\theta^+\chi_+ - \theta^-\chi_-$ is added to the above solution for $\hat{T}$. This, however, does not affect the algebra that follows from that solution.

[5] Note, however, that the commutator of J with $\hat{O}_\pm, \hat{F}$ is always closed provided only that $F_0^2 \neq \kappa_+\kappa_-$.

The corresponding algebras in the above three cases are summarized by their non-trivial relations as follows:

$$\text{CE-SUSY:} \qquad \{Q_+, Q_-\} = c\,, \quad Q_\pm^2 = -P_\pm\,, \tag{3.30}$$

$$\begin{aligned} \text{NC-SUSY:} \qquad \{Q_+, Q_-\} &= \hat{T}\,, & Q_\pm^2 &= -P_\pm\,, \\ [P_\pm, Q_\mp] &= \pm\hat{O}_\pm\,, & [Q_\pm, \hat{T}] &= \pm\hat{O}_\pm\,, \\ \{\hat{O}_\pm, Q_\mp\} &= F_0\,, & [P_+, P_-] &= F_0\,. \end{aligned} \tag{3.31}$$

$$\begin{aligned} \text{LX-SUSY:} \qquad \{Q_+, Q_-\} &= \hat{T}\,, & Q_\pm^2 &= -P_\pm\,, \\ [P_\pm, Q_\mp] &= \pm\hat{O}_\pm\,, & [Q_\pm, \hat{T}] &= \pm\hat{O}_\pm\,, \\ \{\hat{O}_\pm, Q_\pm\} &= \pm i\kappa_\pm\,, & [P_\pm, \hat{T}] &= -i\kappa_\pm\,. \end{aligned} \tag{3.32}$$

It is amusing to note that the last case is a limiting case of the closed algebra for the exponential configuration of $\hat{T}$ found in [14]:

$$\hat{T}(x) = \kappa_+ x^+ + \kappa_- x^- = \lim_{b\to\infty} b\left[\exp\left(\frac{\kappa_+ x^+ + \kappa_- x^-}{b}\right) - 1\right] \tag{3.33}$$

Note that, unless in this particular limit, the exponential configuration is not a solution to the free field EOM of $\mathcal{A}_\alpha$.

4. GSFT with matter

In this section we consider the coupling of a matter sector S to $\mathcal{A}_\alpha$ and investigate the conditions under which it realizes the simplest generalization of the SUSY algebras, namely the CE-SUSY, by its classical EOM. The simplest GSFT model for this field content is obtained by gauging a supersymmetric σ-model for $(S, \bar{S})$ by the U(1) superfield $\mathcal{A}_\alpha$ and adding to it the kinetic term of $\mathcal{A}_\alpha$ together with a U(1) invariant superpotential term. Here we choose a linear σ-model (with a flat target space) and a superpotential which is a function of both $S\bar{S}$ and the superfield strength T of $\mathcal{A}_\alpha$. This gives the following lagrangian density on superspace

$$\begin{aligned} \mathcal{L} &= \frac{1}{4}\mathcal{D}^\alpha T \mathcal{D}_\alpha T + D^\alpha S D_\alpha \bar{S} + W(S\bar{S}, T) \\ &= \frac{1}{2}\mathcal{D}_+ T \mathcal{D}_- T + D_+ S D_- \bar{S} + D_+ \bar{S} D_- S + W(S\bar{S}, T)\,. \end{aligned} \tag{4.1}$$

where $D_\alpha S$, $D_\alpha \bar{S}$ are defined as follows

$$D_\alpha S \equiv (\mathcal{D}_\alpha - i\mathcal{A}_\alpha)S\,, \qquad D_\alpha \bar{S} \equiv (\mathcal{D}_\alpha + i\mathcal{A}_\alpha)\bar{S}\,. \tag{4.2}$$

Except for the coupling of S to $\mathcal{A}_\alpha$ via D_α, the above lagrangian resembles the coupled theory of the real and complex scalar superfields, T, S, respectively. The coupling of S to $\mathcal{A}_\alpha$ comes partly through the gauge supercovariant derivatives in the middle terms and partly from the dependence of the superpotential on $\mathcal{A}_\alpha$ via its dependence on T. The above lagrangian may be written in terms of the component fields as can be read from the expansions of $(S, \mathcal{A}_\pm)$ in the following forms

$$
\begin{aligned}
S(x,\theta) &= \phi(x) + \theta^+\psi_+(x) + \theta^-\psi_-(x) + \theta^+\theta^- D(x)\,, \\
\mathcal{A}_+(x,\theta) &= \xi_+(x) + \theta^+ A_+(x) + \theta^-\sigma_+(x) + \theta^+\theta^-\eta_+(x)\,, \\
\mathcal{A}_-(x,\theta) &= \xi_-(x) + \theta^+\sigma_-(x) + \theta^- A_-(x) + \theta^+\theta^-\eta_-(x)\,.
\end{aligned} \tag{4.3}
$$

From these expansions, we see that the dynamical degrees of freedom in S (the matter multiplet) consist of a complex scalar ϕ and a spinor ψ, while those in $\mathcal{A}_\alpha$ (the gauge multiplet) consist of a complex scalar σ, a spinor ξ and a vector A. The scalar D and the spinor η are non-dynamical fields playing the role of the auxiliary fields. The action and the EOM in the ordinary space can be derived using these expansions after taking the $d^2\theta$ integrations. It is, however, more elegant and convenient to keep the superspace formulation for writing the EOM directly in the language of superfields. Let us denote the variations with respect to $\mathcal{A}_\alpha$ and $(S, \bar{S})$ by $\delta_{\mathcal{A}}$ and δ_S, respectively. Then by varying $\mathcal{L}$ with respect to $\mathcal{A}_\alpha$, after extraction of the total derivative terms, we get the following result

$$
\begin{aligned}
\delta_{\mathcal{A}}\mathcal{L} \cong \delta\mathcal{A}_+ &\left[-i\mathcal{D}_-\left(F + \frac{\partial W}{\partial T}\right) + i(\bar{S}D_-S - SD_-\bar{S})\right] + \\
+\delta\mathcal{A}_- &\left[-i\mathcal{D}_+\left(F + \frac{\partial W}{\partial T}\right) - i(\bar{S}D_+S - SD_+\bar{S})\right],
\end{aligned} \tag{4.4}
$$

where $F \equiv \mathcal{D}_-\mathcal{D}_+T$. This gives the following EOM for $\mathcal{A}_\alpha$

$$
\mathcal{D}_\pm\left(F + \frac{\partial W}{\partial T}\right) = \pm(SD_\pm\bar{S} - \bar{S}D_\pm S)\,. \tag{4.5}
$$

Similarly, variation of $\mathcal{L}$ with respect to $S, \bar{S}$ gives

$$
\delta_S\mathcal{L} \cong \delta S\left(-2D_+D_-\bar{S} - T\bar{S} + \frac{\partial W}{\partial S}\right) + \delta\bar{S}\left(-2D_+D_-S + TS + \frac{\partial W}{\partial \bar{S}}\right), \tag{4.6}
$$

from which the EOM of $S, \bar{S}$ follows

$$
\begin{aligned}
D_+D_-S &= \frac{1}{2}\left(TS + \frac{\partial W}{\partial \bar{S}}\right), \\
D_+D_-\bar{S} &= \frac{1}{2}\left(-T\bar{S} + \frac{\partial W}{\partial S}\right).
\end{aligned} \tag{4.7}
$$

We note that all the above EOM are explicitly written in a gauge covariant way. In particular, the EOM of $\mathcal{A}_\alpha$, eq. (4.5), is a gauge invariant equation with a gauge invariant source on its r.h.s., which is expected to be conserved due to the divergenceless of l.h.s. of the same equation. Indeed, by introducing the spinor supercurrent

$$
\mathcal{J}_\alpha \equiv SD_\alpha\bar{S} - \bar{S}D_\alpha S\,. \tag{4.8}
$$

it is easy to see, using the EOM of $S, \bar{S}$ eqs. (4.7), that

$$
\mathcal{D}^\alpha\mathcal{J}_\alpha = S\frac{\partial W}{\partial S} - \bar{S}\frac{\partial W}{\partial \bar{S}} = 0\,. \tag{4.9}
$$

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The last equality follows from the U(1) invariance of the superpotential W. This conservation equation is a result of the U(1) invariance of the theory defined by eq. (4.1). It is also instructive to note, by inspection of eq. (4.1), that (direct) coupling of $\mathcal{A}_\alpha$ to $(S,\bar{S})$ takes place through the above supercurrent as $i\mathcal{J}^\alpha\mathcal{A}_\alpha$ which is as expected.

4.1 Models allowing a constant T configuration

As we have already mentioned in the introduction, a classical configuration of $\mathcal{A}_\alpha$ possessing a constant superfield strength describes the centrally extended SUSY algebra. It is therefore an interesting question to know under what conditions the above set of EOM allow for a constant T configuration. To find the precise answer to this question, it is more convenient to work with the polar coordinates (R,Φ) on the plane of the complex coordinates $(S,\bar{S})$,

$$S = Re^{i\Phi}, \qquad \bar{S} = Re^{-i\Phi}\,. \tag{4.10}$$

The lagrangian (4.1) in terms of R, Φ, $\mathcal{A}_\pm$ then becomes

$$\mathcal{L} = \mathcal{D}_+R\mathcal{D}_-R + R^2(\mathcal{D}_+\Phi - \mathcal{A}_+)(\mathcal{D}_-\Phi - \mathcal{A}_-) + \frac{1}{2}\mathcal{D}_+T\mathcal{D}_-T + W(R,T). \tag{4.11}$$

The EOM of $R,\Phi,\mathcal{A}_\pm$ following from this lagrangian are

$$\begin{aligned}
\mathcal{D}_+\mathcal{D}_-R - R(\mathcal{D}_+\Phi - \mathcal{A}_+)(\mathcal{D}_-\Phi - \mathcal{A}_-) &= \frac{1}{2}\frac{\partial W}{\partial R}\,,\\
\mathcal{D}_+\left(R^2(\mathcal{D}_-\Phi - \mathcal{A}_-)\right) - \mathcal{D}_-\left(R^2(\mathcal{D}_+\Phi - \mathcal{A}_+)\right) &= 0\,,\\
\mathcal{D}_\pm\left(F + \frac{\partial W}{\partial T}\right) \pm iR^2(\mathcal{D}_\pm\Phi - \mathcal{A}_\pm) &= 0\,.
\end{aligned} \tag{4.12}$$

The second of these equations (the EOM of Φ) is easily integrated to

$$R^2(\mathcal{D}_\pm\Phi - \mathcal{A}_\pm) = \pm i\mathcal{D}_\pm\Omega\,, \tag{4.13}$$

where Ω is any arbitrary real scalar superfield. Putting this result into the third eq. (4.12) (the EOM of $\mathcal{A}_\pm$) then gives $\mathcal{D}_\pm\left(F + \frac{\partial W}{\partial T} - \Omega\right) = 0$, which once again is integrated to[6]

$$\Omega = F + \frac{\partial W}{\partial T}\,. \tag{4.14}$$

Also using the same eq. (4.13) in the first eq. (4.12) (the EOM of R) gives

$$\mathcal{D}_+\mathcal{D}_-R - \frac{1}{R^3}\mathcal{D}_+\Omega\mathcal{D}_-\Omega = \frac{1}{2}\frac{\partial W}{\partial R}\,. \tag{4.15}$$

On the other hand, the eq. (4.13) is solved for $\mathcal{A}_\pm$ to

$$\mathcal{A}_\pm = \mp i\frac{\mathcal{D}_\pm\Omega}{R^2} + \mathcal{D}_\pm\Phi\,, \tag{4.16}$$

[6] Actually, a constant term on the r.h.s. of the eq. (4.14) is allowed. However, one can reabsorb such a constant α in the definition of $W(R,T)$ by adding to it a linear term αT.

using which T is determined in terms of R, Ω as follows

$$\begin{aligned} T &= -i\mathcal{D}_+ \left(i\frac{\mathcal{D}_-\Omega}{R^2} \right) - i\mathcal{D}_- \left(-i\frac{\mathcal{D}_+\Omega}{R^2} \right) \\ &= \frac{2}{R^2}\mathcal{D}_+\mathcal{D}_-\Omega - \frac{2}{R^3}(\mathcal{D}_+R\mathcal{D}_-\Omega + \mathcal{D}_+\Omega\mathcal{D}_-R) \,. \end{aligned} \tag{4.17}$$

Note that the Φ dependence disappears as a result of the gauge independence of T.

Let us now focus on a constant T configuration. Since in this case $F = 0$, by the eq. (4.14), Ω can be written as a function of (R, T) as follows

$$\Omega(R,T) = \frac{\partial W}{\partial T} \,. \tag{4.18}$$

Taking hence $\Omega = \Omega(R, T)$ in the previous expression for T, after simple manipulations, we are led to the following equation for R

$$\frac{1}{R^2}\frac{\partial\Omega}{\partial R}\mathcal{D}_+\mathcal{D}_-R + \frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial\Omega}{\partial R}\right)\mathcal{D}_+R\mathcal{D}_-R = \frac{1}{2}T \,, \tag{4.19}$$

in which T must be interpreted as a constant parameter. This should be compared to the EOM of R, eq. (4.15) , which upon taking $\Omega = \Omega(R, T)$, is written as

$$\mathcal{D}_+\mathcal{D}_-R - \frac{1}{R^3}\left(\frac{\partial\Omega}{\partial R}\right)^2\mathcal{D}_+R\mathcal{D}_-R = \frac{1}{2}\frac{\partial W}{\partial R} \,. \tag{4.20}$$

Each of the eqs. (4.19) ,(4.20) is in the form of a differential equation governing the same variable $R(x, \theta)$. For being identical equations, their coefficients as functions of (R, T) must be proportional. This gives the following conditions

$$\begin{aligned} \frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial\Omega}{\partial R}\right) &= -\frac{1}{R^5}\left(\frac{\partial\Omega}{\partial R}\right)^3 , \\ \frac{\partial W}{\partial R} &= T\left(\frac{1}{R^2}\frac{\partial\Omega}{\partial R}\right)^{-1} \,. \end{aligned} \tag{4.21}$$

Owing to $\Omega = \partial W/\partial T$, the above conditions are written in the form of differential equations governing $W(R, T)$. The first of the eqs. (4.21) is easily integrated for $\partial\Omega/\partial R$ to

$$\frac{\partial\Omega}{\partial R} = \frac{R^2}{\sqrt{R^2 + c(T)}} \,, \tag{4.22}$$

where $c(T)$ is a yet arbitrary function and a $\pm$ choice of sign behind the square root is understood. We can determine $c(T)$ using the consistency condition between this equation and the second eq. (4.21) which may now be written as

$$\frac{\partial W}{\partial R} = T\sqrt{R^2 + c(T)} \,. \tag{4.23}$$

Taking $\partial/\partial T$ of this expression and equating it to the r.h.s. of eq. (4.22) gives the following integrability condition

$$c(T) + \frac{1}{2}T\frac{\partial c(T)}{\partial T} = 0 \qquad \Rightarrow \qquad c(T) = \frac{a}{T^2} \,, \tag{4.24}$$

with a being any real constant. Hence we find

$$\frac{\partial W}{\partial R} = \sqrt{T^2R^2+a}\,. \tag{4.25}$$

The general solution of this equation for $W(R,T)$ depending on the sign of a is written as

$$\begin{aligned}
W(R,T) &= \frac{1}{2}\left(\frac{a}{T}\sinh^{-1}\frac{TR}{\sqrt{a}}+R\sqrt{T^2R^2+a}\right)+w(T)\,; && a>0\,,\\
W(R,T) &= \frac{1}{2}\left(\frac{|a|}{T}\cosh^{-1}\frac{TR}{\sqrt{|a|}}+R\sqrt{T^2R^2-|a|}\right)+w(T)\,; && a<0\,,\\
W(R,T) &= \frac{1}{2}TR^2+w(T)\,; && a=0\,,
\end{aligned} \tag{4.26}$$

where $w(T)$ is an arbitrary real function. Note that in the $a<0$ case, (R,T) must obey the restriction $TR \geq |a|$. These form of the superpotential (with $R^2 = S\bar{S}$) guarantee the existence of a constant T configuration allowed by the EOM of the original theory, eqs. (4.12).

4.2 An effective lagrangian for the general constant T configurations

It is useful to find an effective lagrangian formulation of the EOM of R in which all the other degrees of freedom $(\Phi, \mathcal{A}_\alpha)$ are eliminated while T is treated as a constant parameter. Such formulation is achieved by assuming an effective lagrangian of the form

$$\mathcal{L}_{\text{eff}}[R] = \frac{1}{2}G(R,T)\mathcal{D}_+R\,\mathcal{D}_-R + W_{\text{eff}}(R,T) \tag{4.27}$$

and demanding that its EOM for R is identical to eq. (4.19) for R. This fixes the functions $G, \partial W_{\text{eff}}/\partial R$ as follows

$$\begin{aligned}
G &= \left(\frac{1}{R^2}\frac{\partial\Omega}{\partial R}\right)^2 = \frac{1}{R^2+a/T^2}\,,\\
\frac{\partial W_{\text{eff}}}{\partial R} &= \frac{1}{2}\frac{T}{R^2}\frac{\partial \Omega}{\partial R} = \frac{T}{2\sqrt{R^2+a/T^2}}\,.
\end{aligned} \tag{4.28}$$

The effective superpotential for R thus depending on a becomes

$$\begin{aligned}
W_{\text{eff}}(R,T) &= \frac{1}{2}T\sinh^{-1}\left(\frac{TR}{\sqrt{a}}\right); && a>0\,,\\
W_{\text{eff}}(R,T) &= \frac{1}{2}T\cosh^{-1}\left(\frac{TR}{\sqrt{|a|}}\right); && a<0\,,\\
W_{\text{eff}}(R,T) &= \frac{1}{2}T\,\ln R\,; && a=0\,.
\end{aligned} \tag{4.29}$$

The lagrangian (4.27) can be put into a very simple form by changing the variable R to $\hat{R} = \hat{R}(R,T)$ which brings its kinetic part into the standard form $\frac{1}{2}\mathcal{D}_+\hat{R}\,\mathcal{D}_-\hat{R}$. The function $\hat{R}(R,T)$ hence must be such that

$$\frac{\partial \hat{R}}{\partial R} = \sqrt{G} = \frac{1}{\sqrt{R^2+a/T^2}}\,. \tag{4.30}$$

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By comparison to the eq. (4.28), the solution for $\hat{R}$ (apart from an irrelevant T-dependent shift) is essentially the same as W_{eff} divided by $\frac{1}{2}T$. So the effective superpotential in terms of the variable $\hat{R}$ becomes $W_{\text{eff}}(\hat{R},T) = \frac{1}{2}T\hat{R}$. The $\mathcal{L}_{\text{eff}}[\hat{R}]$ thus becomes

$$\mathcal{L}_{\text{eff}}[\hat{R}] = \frac{1}{2}\left(\mathcal{D}_+\hat{R}\mathcal{D}_-\hat{R} + T\hat{R}\right). \tag{4.31}$$

This gives a simple EOM for $\hat{R}$ of the form

$$\mathcal{D}_+\mathcal{D}_-\hat{R} = \frac{1}{2}T. \tag{4.32}$$

In analogy to the eq. (3.20), the general solution of this equation for $\hat{R}$ is written as

$$\hat{R}(x,\theta) = \hat{\phi}_+(x^+) + \hat{\phi}_-(x^-) + \theta^+\hat{\psi}_+(x^+) + \theta^-\hat{\psi}_-(x^-) - \frac{1}{2}T\theta^+\theta^- . \tag{4.33}$$

where $\hat{\phi}_\pm(x^\pm), \hat{\psi}_\pm(x^\pm)$ are arbitrary holomorphic functions. The general solution for $R(x,\theta)$ then easily follows from the inverse of the equation $\hat{R} = \hat{R}(R,T)$ for R in terms of $(\hat{R},T)$. For example, in the case of $a = 0$, with $R = \mathrm{e}^{\hat{R}}$, we find the following solution

$$R(x,\theta) = \mathrm{e}^{\hat{\phi}_+(x^+)+\hat{\phi}_-(x^-)}\begin{bmatrix} 1 + \theta^+\hat{\psi}_+(x^+) + \theta^-\hat{\psi}_-(x^-) \\ -\theta^+\theta^-\left(\frac{1}{2}T + \hat{\psi}_+(x^+)\hat{\psi}_-(x^-)\right)\end{bmatrix}. \tag{4.34}$$

This indeed specifies all the constant T non-BPS solutions of the model of eq. (4.11) with a superpotential of the form of eqs. (4.26) in the case $a = 0$. The BPS solutions for more general superpotentials are studied in the following section.

5. BPS states

It is well known that central charges in superalgebras are realized as topological charges of the BPS solitons in supersymmetric field theories realizing these algebras. These solutions are specified by their distinguished property that they preserve a specific amount of the underlying SUSY of the theory [20, 21, 22]. For a supersymmetric gauge theory (like our GSFT model) the SUSY preservation condition for a field configuration means that, for specific choices of the SUSY parameters, the SUSY variations of all the *gauge invariant* degrees of freedom for that configuration are vanishing. For the model at hand, the gauge invariant degrees of freedom encoded in our superfield variables, $S, \bar{S}, \mathcal{A}_\pm$, can be summarized in the two scalar superfields R, T together with a spinor superfield defined by $(\mathcal{D}_\alpha\Phi - \mathcal{A}_\alpha)$. On the EOM, however, we have seen that these reduce to only the three quantities (R, T, Ω). As such, the supersymmetric configurations of the GSFT model can be specified as those satisfying the following conditions

$$\delta_\epsilon R = 0\,, \qquad \delta_\epsilon T = 0\,, \qquad \delta_\epsilon \Omega = 0\,, \tag{5.1}$$

for some specific choices of the SUSY parameters $\epsilon^\pm$. On the other hand, the three superfields (R, T, Ω) are totally governed by the closed set of eqs. (4.14), (4.15), (4.17) which may now be interpreted as their set of second order EOM:

$$
\begin{aligned}
\mathcal{D}_+\mathcal{D}_-R - \frac{1}{R^3}\mathcal{D}_+\Omega\mathcal{D}_-\Omega - \frac{1}{2}\frac{\partial W}{\partial R} &= 0\,,\\
\mathcal{D}_+\mathcal{D}_-\Omega - \frac{1}{R}(\mathcal{D}_+R\mathcal{D}_-\Omega + \mathcal{D}_+\Omega\mathcal{D}_-R) - \frac{1}{2}TR^2 &= 0\,,\\
\mathcal{D}_+\mathcal{D}_-T - \frac{\partial W}{\partial T} + \Omega &= 0\,.
\end{aligned}
\tag{5.2}
$$

It is then easy to find that these set of EOM are generated by the following "effective lagrangian":

$$
\mathcal{L}_{\rm eff}[R,\Omega,T] = \mathcal{D}_+R\mathcal{D}_-R - \frac{1}{R^2}\mathcal{D}_+\Omega\mathcal{D}_-\Omega + \frac{1}{2}\mathcal{D}_+T\mathcal{D}_-T + W(R,T) - \Omega T\,. \tag{5.3}
$$

This is in the form of a supersymmetric non-linear σ-model which is coupled to a superpotential term (see [22] for a general 4D version of σ-models). The target space metric of this model is given by

$$
ds^2 = dT^2 + 2dR^2 - \frac{2}{R^2}d\Omega^2\,, \tag{5.4}
$$

while its superpotential is defined as follows

$$
W_{\rm eff}(R,\Omega\,,T) \equiv W(R,T) - \Omega T\,. \tag{5.5}
$$

For a generic non-linear σ-model with a general metric of euclidean signature and an arbitrary superpotential with several extrema, it can be shown that there are always finite energy BPS solutions of the EOM that minimize the energy functional of the model [21]. These BPS solutions are governed by first order differential equations and can be shown to preserve some fraction (say 1/2 in our case) of the total supersymmetries that underlie the model (see appendix). For the case at hand, although the signature of the target space metric is lorentzian, we can still define 1/2 SUSY solutions using the same first order equations as that of the BPS solutions. These then will specify BPS states of the original GSFT model because the lagrangian of this model, eq. (4.1), in terms of the variables $(R,\Omega\,,T)$ is written as a σ-model with a metric of euclidean signature:

$$
\mathcal{L}[R,\Omega\,,T] = \mathcal{D}_+R\mathcal{D}_-R + \frac{1}{R^2}\mathcal{D}_+\Omega\mathcal{D}_-\Omega + \frac{1}{2}\mathcal{D}_+T\mathcal{D}_-T + W(R,T)\,. \tag{5.6}
$$

The BPS configurations of the GSFT model are so the 1/2 SUSY solutions of the σ-model (5.3) which are given by the following equations (see the general eq. (A.15)):

$$
\begin{aligned}
\pm\frac{dR}{dx} &= \frac{1}{2}\frac{\partial W}{\partial R}\,,\\
\pm\frac{d\Omega}{dx} &= \frac{1}{2}TR^2\,,\\
\pm\frac{dT}{dx} &= \frac{\partial W}{\partial T} - \Omega\,.
\end{aligned}
\tag{5.7}
$$

The $\pm$ signs on the r.h.s. are the same for all these three equations and they distinguish between the soliton and anti-soliton (kink and anti-kink) solutions. Actually, we must consider these as the set of equations governing the scalar components of R, Ω, T, say $\phi_R, \phi_\Omega, \phi_T$. As in the appendix, however, it can be shown that they are also satisfied by the whole superfields for the BPS states involving their auxiliary field components D_R, D_Ω, D_T, with their spinorial components set to zero (eq. (A.24)). In the following, we take this facility to simplify our notation in all the subsequent equations.

Along the line of our search in this paper, it is now natural to ask whether there are BPS solutions possessing a constant T configuration. Form the third eq. (5.7) it is clear that the existence of such solutions requires

$$\frac{dT}{dx} = 0 \qquad \Rightarrow \qquad \Omega = \Omega(R,T) = \frac{\partial W}{\partial T}\,. \tag{5.8}$$

The consistency conditions for the existence of such configurations is found from $\frac{d\Omega}{dx} = \frac{\partial\Omega}{\partial R}\frac{dR}{dx}$ and the first two of the eqs. (5.7) to be

$$\frac{\partial W}{\partial R} = T\left(\frac{1}{R^2}\frac{\partial\Omega}{\partial R}\right)^{-1}. \tag{5.9}$$

This is just one of the consistency conditions (4.21) for the EOM to allow for a constant T configuration. The reason for missing the other condition is simply that the term proportional to $\mathcal{D}_+ R\mathcal{D}_- R$ in both equations (4.19), (4.20) is vanishing on a BPS solution and, hence, the proportionality of the corresponding coefficients is not required as for their other terms. Owing to $\Omega(R,T) = \partial W/\partial T$, we may rewrite the above condition as

$$\frac{1}{2}\frac{\partial}{\partial T}\left(\frac{\partial W}{\partial R}\right)^2 = TR^2\,, \tag{5.10}$$

which can be integrated to

$$\left(\frac{\partial W}{\partial R}\right)^2 - T^2R^2 = f(R)\,, \tag{5.11}$$

with $f(R)$ being an arbitrary real function of R. We note that the result of the analysis from the consistency conditions with the EOM, eq. (4.25), is a particular case of the above result in which $f(R) = a$ is a constant. The above equation is once again integrated to

$$W(R,T) = w(T) + \int dR\sqrt{T^2R^2 + f(R)}\,. \tag{5.12}$$

From this expression, we see that generally the function $W(R,T)$ possesses branch points. The physical solutions of the theory are determined by the regions between these branch points where under the square root is always positive or zero. These solutions are BPS states only under certain circumstances as will be explained in the next subsection.

Under the above condition, the BPS states of the GSFT with a constant value of T are identical to the BPS states of the effective Wess-Zumino model defined as

$$\mathcal{L}_{WZ}[R] = \mathcal{D}_+ R\mathcal{D}_- R + W(R,T) \qquad (T = \text{const})\,. \tag{5.13}$$

This can be shown easily by comparing the BPS equation and the EOM of this model to those of the σ-model in eq. (5.3). Owing to the general equation (A.25), the energy functional for the BPS configurations then reduces to

$$\mathcal{M} = \int dx d^2\theta\, \mathcal{L}[R,T,\Omega] = \int dx d^2\theta\, W(R,T)\,. \tag{5.14}$$

For the "pure" BPS states with a net U(1) charge (see next subsection), this mass formula is further simplified to

$$\mathcal{M}(T) = W(R_\infty(T),T) - W(R_{-\infty}(T),T)\,, \tag{5.15}$$

where $R_\infty(T)$ and $R_{-\infty}(T)$ stand for the boundary values of R lying at the branch points of W. The BPS condition then requires that $\mathcal{M}(T)$ equals one half the central charge $\mathcal{Z}(T)$ of the corresponding superalgebra [20] (see also appendix), which for dimensional reasons must be written as a multiple of T as follows

$$\mathcal{M}(T) = \frac{1}{2}\mathcal{Z}(T) = \frac{1}{2} gT\,. \tag{5.16}$$

Here, the dimensionless quantity g measures the U(1) charge of the soliton with respect to the gauge superfield $\mathcal{A}_\alpha$. The reason for identifying g as the U(1) charge of the "solitonic state" of mass $\mathcal{M}(T)$, in contrast to charge e of the "elementary states", is that it appears on equal footing to e on the r.h.s. of the anticommutator of the supercharges (compare to the first eq. (2.4)),

$$\{Q_+, Q_-\} = gT\,. \tag{5.17}$$

Further evidence for the interpretation of g as a U(1) charge comes from specifying the coupling of $\mathcal{A}_\alpha$ to a BPS soliton via studying its low energy dynamics taking into account its fermionic zero modes [15].

Using the above result for W, the mass formula of the pure BPS states becomes

$$\mathcal{M}(T) = \int_{R_{-\infty}(T)}^{R_\infty(T)} dR\sqrt{T^2R^2 + f(R)}\,. \tag{5.18}$$

The boundary values $R_{-\infty}(T)$, $R_\infty(T)$ are two real roots of $\partial W/\partial R$ or branch points of W hence satisfying

$$T^2R^2 + f(R) = 0\,. \tag{5.19}$$

The BPS condition then fixes the value of T in terms of g (the U(1) charge of the soliton)

$$\mathcal{M}(T) = \frac{1}{2} gT \qquad \Rightarrow \qquad T = T(g)\,. \tag{5.20}$$

In view of the fact that the BPS states of the GSFT model are solutions to the effective lagrangian (5.3), one is tempted to introduce an "effective mass" $\mathcal{M}_{\text{eff}}$, which in contrast to the "real mass" $\mathcal{M}$, is defined by the superspace integral of $W_{\text{eff}}(R,T,\Omega)$ instead of $W(R,T)$. Now, by eqs. (5.5), (5.8) for constant T configurations, W_{eff} is written as a Legendre transformation of the ordinary superpotential W as follows

$$W_{\text{eff}} = W(R,T) - T\frac{\partial W(R,T)}{\partial T}\,. \tag{5.21}$$

Taking the integral of both sides over superspace, and using an expression like eq. (5.15) for $\mathcal{M}_{\text{eff}}$, we find a similar Legendre transformation between the effective and the real masses as follows

$$\mathcal{M}_{\text{eff}}(T) = \mathcal{M}(T) - T\frac{\partial \mathcal{M}(T)}{\partial T}\,. \tag{5.22}$$

Finally, using eq. (5.16) in this result, considering g as a function of T , gives the "effective U(1) charge" g_{eff} as

$$g_{\text{eff}}(T) = -T\frac{\partial g(T)}{\partial T}\,. \tag{5.23}$$

That is to say, the central charge as seen by the effective theory (5.3) is given by $\mathcal{Z}_{\text{eff}}(T) = T\,g_{\text{eff}}(T)$. In particular, for the elementary states of the GSFT model, with $g = e = \pm 1$, we find $\mathcal{Z}_{\text{eff}} = g_{\text{eff}} = 0$ which is a plausible result as neither of the states R, T, Ω in the effective theory carry a non-zero U(1) charge.

5.1 Types of solitons (pure versus dipole charges)

As was mentioned above, on the space of constant T configurations, the solitonic solutions are described by BPS solutions of the WZ model in eq. (5.13). For any constant value of T, the "effective potential" of this theory is given by the following expression:

$$V(R,T) = \frac{1}{4}\left(\frac{\partial W}{\partial R}\right)^2 = \frac{1}{4}\left[T^2R^2 + f(R)\right]. \tag{5.24}$$

The most general solitonic solutions of this theory are then governed by the following first order equation

$$\left(\frac{dR}{dx}\right)^2 = V(R,T)\,, \tag{5.25}$$

which is nothing but the square of the first eq. (5.7). By interpreting x as a time variable, eq. (5.25) resembles the energy conservation equation for motion of a particle of vanishing energy in the R direction under the influence of a force derived from the potential $-\frac{1}{2}V(R,T)$. Physically allowed regions of "motion" are so determined by the values of R satisfying

$$V(R,T) \geq 0\,. \tag{5.26}$$

Let us assume that $V(R,T)$ is not an always positive or negative function of (R,T). There are so several real roots of $V(R,T) = 0$ at $R = R_1(T), R_2(T)$, etc. Take two successive of these roots, say $R_1(T), R_2(T)$, between which $V(R,T)$ is always positive. It is clear that if $W(R,T)$ were a single valued analytic function of R, $V(R,T)$ would vanish along with its derivative $\partial V/\partial R$ at all its roots. Such points would then determine the SUSY vacua of the theory and the BPS solitons were in the form of kink and anti-kink solitons [23] interpolating between different such vacua at $x = -\infty$ and $x = +\infty$. Such solutions would preserve a *fixed half* of SUSY all the way between $-\infty < x < \infty$. However, there may occur different situations when $W(R,T)$ is not a single valued function like the case in eq. (5.12). Then, in general, we can distinguish three types of solutions of eq. (5.25) depending on $\partial V/\partial R$ at the above roots of V:

(i) The first possibility is that $\partial V/\partial R = 0$ at both the points $R = R_1(T), R_2(T)$. Such points correspond to supersymmetric vacua of our original theory defined by eq. (4.1) which may be achieved *only* for particular values of T. Any such vacuum (if any) is determined by a common solution of the following two equations for (R, T):

$$\begin{aligned} T^2R^2 + f(R) &= 0\,, \\ 2T^2R + f'(R) &= 0\,. \end{aligned} \tag{5.27}$$

The soliton interpolating between R_1, R_2 from these equations is then a BPS state in the form of a supersymmetric kink or anti-kink solution extending from $x = -\infty$ to $x = +\infty$ and preserving a *fixed* half of SUSY in between. Such solutions possess a definite mass and a net U(1) charge as given in the previous subsection.

(ii) Another possibility is that $\partial V/\partial R = 0$ only at one of the two roots, say at the smaller one $R_1(T)$. The solution for $R(x)$ is then in the form of a curve with a hump of height $R_2(T)$ at $x = 0$ and flat tails of height $R_1(T)$ at $x = \pm\infty$. Since dR/dx changes sign at $x = 0$, the two halves of the solution at $-\infty < x < 0$ and $0 < x < \infty$ lie on different branches of the kink/anti-kink equation $dR/dx = \pm\partial W/\partial R$ and, hence, the solution preserves *different halves* of SUSY in its two halves. From the general expression (5.14) with the first eq. (5.7), this solution has a finite mass of the form

$$\mathcal{M}(T) = 2\int_{R_1(T)}^{R_2(T)} dR\sqrt{T^2R^2 + f(R)}\,, \tag{5.28}$$

while its central charge $\mathcal{Z}(T) = W(R_\infty(T), T) - W(R_{-\infty}(T), T)$ vanishes. In the light of our previous interpretation of the central charge, this means that the net U(1) charge g of such solutions is vanishing and hence they must be interpreted as the U(1) "dipole charges" or bound states of the kink/anti-kink solutions.

(iii) For a general value of T, $\partial V/\partial R$ does not vanish along with V at these roots and the solution for $R(x)$ is in the form of a periodic function of x oscillating between $R_1(T), R_2(T)$ with a period

$$L(T) = 4\int_{R_1(T)}^{R_2(T)} \frac{dR}{\sqrt{T^2R^2 + f(R)}}\,. \tag{5.29}$$

If the coordinate x is assumed to be non-compact (i.e. $-\infty < x < \infty$), the above solution corresponds to a periodic array of solitons with separations L along the x direction. Since dR/dx changes sign once at each period, R must satisfy one of the kink/anti-kink equations $dR/dx = \pm\partial W/\partial R$ in either of its half periods and, as such, it must preserve different halves of SUSY in different halves of its each period. The central charge of the algebra corresponding to this solution is also vanishing while its portion of mass along each period is given by the same eq. (5.28). Such solutions may be interpreted as a periodic array of dipole charges or, alternatively, as a single dipole on a circle of radius $L(T)/2\pi$. We note, as is expected, that $L(T) \to \infty$ for the solutions in each of the previous two cases.

The above mentioned points are illustrated by the following three examples.

Example 1. Let us consider as the first example the case of a constant $f(R)$ which followed as the consistency condition with the EOM (see eq. (4.25)):

$$f(R) = a \qquad \Rightarrow \qquad \frac{\partial W}{\partial R} = \sqrt{T^2R^2 + a}\,. \tag{5.30}$$

Clearly, only for $a \leq 0$, there are real roots of $\frac{\partial W}{\partial R}$ lying at $R = \pm\frac{\sqrt{-a}}{T}$. Even in this case, there are no BPS solitons of the model for any constant value of T, since between the two roots always we have $V < 0$.

Example 2. We now consider the case:

$$f(R) = -\alpha^2 R^4 \qquad \Rightarrow \qquad \frac{\partial W}{\partial R} = \sqrt{T^2R^2 - \alpha^2 R^4}\,. \tag{5.31}$$

The roots of $\partial W/\partial R$ in this case are at $R = 0, \pm\beta$, where $\beta \equiv T/\alpha$. Only $R = 0$ describes a supersymmetric vacuum point as only at this point both $V = 0$, $\partial V/\partial R = 0$ hold. Since between each pair of these roots, $V > 0$, there are physical solutions with a constant value of T. However, that $\partial V/\partial R$ does vanish only at $R = 0$ but not at the other two roots means that the solution for $R(x)$ is a symmetric function of x extending from $x = -\infty$ to $x = \infty$, going into the vacuum $R = 0$ at both extremes and passing through the maximum $R = \beta$ or minimum $R = -\beta$ at the middle point $x = 0$. The mass of each of these solutions is given by

$$\mathcal{M}(T) = 2\int_0^{T/\alpha} dR\sqrt{T^2R^2 - \alpha^2R^4} = \frac{2T^3}{3\alpha^2}\,. \tag{5.32}$$

Such solutions describe a domain wall that carries the "dipole charge" of the U(1) gauge superfield $\mathcal{A}_\alpha$ but its net charge is vanishing. The explicit form of the superpotential in this case is given by

$$W(R,T) = w(T) - \frac{\alpha}{3}\left(\frac{T^2}{\alpha^2} - R^2\right)^{\frac{3}{2}}\,. \tag{5.33}$$

Example 3. As the last example, we derive the form of the superpotential $W(R,T)$ that for a particular value of T leads to the superpotential of the sine-Gordon model:

$$W_0(R) = \alpha \sin R\,. \tag{5.34}$$

For this model, common roots of $V, \partial V/\partial R$ lie at $R_n^{(0)} = (2n+1)\frac{\pi}{2}$, $n \in \mathbf{Z}$. Hence each two successive of these roots (say $\pm\frac{\pi}{2}$) define a BPS soliton of mass

$$\mathcal{M} = \alpha \sin R\,|_{R=-\pi/2}^{\pi/2} = 2\alpha\,. \tag{5.35}$$

The BPS condition, eq. (5.16), then fixes the particular value of T in terms of the U(1) charge g of the soliton as follows

$$T_0 = \frac{4\alpha}{g}\,. \tag{5.36}$$

Using this value of T in eq. (5.11), we find the appropriate form of $f(R)$,

$$f(R) = \alpha^2\left(\cos^2 R - \frac{4}{g^2}R^2\right)\,. \tag{5.37}$$

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This using eq. (5.12) fixes the appropriate superpotential of the GSFT model (4.1) for the sine-Gordon model to be consistently embedded into this model:

$$W(R,T) = w(T) + \alpha \int dR \sqrt{\cos^2 R + \left(\frac{T^2}{\alpha^2} - \frac{16}{g^2}\right) R^2}\,. \tag{5.38}$$

As can be seen by sketching the curve of V in terms of R for a fixed value of T, we distinguish the following three regimes of the solitonic solutions (for simplicity we assume $T > 0$):

(a) $T > T_0$, there are no BPS solutions with a finite mass,

(b) $T = T_0$, there are BPS solutions interpolating between $R_n^{(0)}, R_{n+1}^{(0)}$,

(c) $T < T_0$, one finds several discrete values of T,

$$\begin{aligned} T_n &= \sqrt{T_0^2 - \sin^2 R_n}\,, \\ R_n &= -\cot R_n; \qquad (n = \pm 1, \pm 2, \ldots)\,, \end{aligned} \tag{5.39}$$

so that at points $R = R_n$, $T = T_n$ both V and $\partial V/\partial R$ vanish. These form a set of "frozen" isolated vacua of the theory that solve the EOM but do not allow any fluctuations of the fields in their vicinity. Between $T_n < T < T_{n+1}$, we find a finite number of non-BPS periodic solutions that correspond to arrays of the dipole charges. As T is increased to pass one of the isolated vacua at $T = T_n$, the number of such solutions is increased by two. As $n \to \infty$, we find $R_n \to n\pi$ and $T_n \to T_0$ and so the BPS solutions in case (b) are recovered.

6. Conclusion

In this paper we studied a specific model of U(1) GSFT with matter — namely the supersymmetric gauged σ-model in $D = 2$, $\mathcal{N} = 1$ superspace — as a model realizing the generalized noncommutative superalgebra of ref. [14] as an algebra underlying its symmetries. We found the most general solutions to the free field EOM of the gauge superfield $\mathcal{A}_\alpha$ that lead to such closed superalgebras. It was shown that, in addition to the usual isotropic solutions corresponding to the centrally extended and ordinary noncommutative supersymmetry algebras, there is also another anisotropic solution that leads to a closed non SO(2) invariant generalization of the ordinary super E^2 algebra.

We then focused on configurations of $\mathcal{A}_\alpha$ with a constant superfield strength T which are allowed by the full EOM of the theory involving those of matter superfield S and its conjugate $\bar{S}$, and derived the conditions under which such configurations are allowed. For general solutions with an arbitrary constant T, these conditions were found that fix the superpotential of the model as the function $W(R,T)$, where $R = (S\bar{S})^{1/2}$, provided we start with a flat metric on the target manifold defined by $(S, \bar{S})$. We obtained an effective theory of the superfield R, with all the gauge degrees of freedom eliminated, when considering the superfield strength T to be a constant. We established that, with the above fixed form of

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the effective superpotential, the effective theory of R after a change of variable $\hat{R} = \hat{R}(R, T)$ is given by a Wess-Zumino model with a linear superpotential term $\frac{1}{2}T\hat{R}$. This led to the general solution for $R(x, \theta)$ on the space of constant T configurations.

We then considered the BPS solutions of the GSFT model and showed that they are generally described by the BPS states of a supersymmetric nonlinear σ-model with a specific target space metric and a particular superpotential which is somehow related to $W(R, T)$. We observed that the condition for the existence of BPS configurations with a constant value of T restricts the form of $W(R, T)$ up to an arbitrary real function $f(R)$ in its expression. (An additive function of T is also allowed but it is not relevant for the purpose of this paper.) Via the analysis of the 1/2 BPS solitonic solutions of this effective theory with a constant value of T, we related their mass and topological charge to the classical value of T and interpreted the topological charge as a U(1) charge with respect to $\mathcal{A}_\alpha$. We argued that, due to the non-single valued form of $W(R, T)$ for a generic $f(R)$ and T, pure BPS solutions with a net U(1) charge and finite mass generally do not exist. Generically, we find a single dipole or an array of dipole charges distributed periodically over one direction. These solutions are not BPS in the usual sense and carry no net U(1) charge. Only for particular forms of $f(R)$ and/or values of T, one finds pure charges corresponding to 1/2 BPS solutions satisfying the usual BPS condition: $\mathcal{M} = \frac{1}{2}|\mathcal{Z}|$. We illustrated these several situations by a few number of simple examples.

A. BPS states of a generic nonlinear σ-model in 2 dimensions

In this appendix we analyze BPS solutions of a generic supersymmetric σ-model with $\mathcal{N} = 1$ SUSY in 2 dimensions which is supplemented by a superpotential term.[7] The field content of this model can be summarized in a number of real scalar superfields $R^i(x, \theta)$, $i = 1, \ldots, n$, with the following θ-expansions

$$R^i(x, \theta) = \phi^i(x) + \bar{\theta}\psi^i(x) + \frac{1}{2}\bar{\theta}\theta D^i(x)\,. \tag{A.1}$$

In these expansions, ϕ^i, ψ^i are the scalar and spinor fields in a hypermultiplet while D^i denotes the corresponding auxiliary field. The superspace action of the model is then written in the following form

$$I[R] = \int d^2x d^2\theta \left(\frac{1}{2}G_{ij}(R)\mathcal{D}_+ R^i \mathcal{D}_- R^j + W(R)\right). \tag{A.2}$$

Here $G_{ij}(R)$ is the metric of the target manifold parametrized by R^i which is assumed to be of euclidean signature and $W(R)$ is a well behaved superpotential term with several extrema. The ordinary generalized Wess-Zumino model is a special case of this σ-model in which the target space is endowed with the flat metric $G_{ij} = \delta_{ij}$. Using the above expansions of R^i and carrying the θ-integrations in this action then gives the lagrangian

[7] The material in this appendix is a 2D version of what presented in [20, 21, 22] which is adapted to our needs throughout this paper.

density of the component fields:

$$
\begin{aligned}
\mathcal{L}[\phi,\psi,D] = & \frac{1}{2}G_{ij}(\phi)\left(\partial_+\phi^i\partial_-\phi^j - i\psi^{i+}\partial_+\psi^{j+} - i\psi^{i-}\partial_-\psi^{j-} - D^iD^j\right) - \\
& -\frac{1}{2}\left(\frac{\partial G_{ik}}{\partial\phi^j} + \frac{\partial G_{jk}}{\partial\phi^i} - \frac{\partial G_{ij}}{\partial\phi^k}\right)\psi^{i-}\psi^{j+}D^k - \frac{\partial W}{\partial\phi^k}D^k - \\
& -\frac{i}{2}\frac{\partial G_{ij}}{\partial\phi^k}\left(\partial_+\phi^i\psi^{j+}\psi^{k+} + \partial_-\phi^i\psi^{j-}\psi^{k-}\right) + \\
& +\frac{1}{2}\frac{\partial^2 G_{ij}}{\partial\phi^k\partial\phi^l}\psi^{i-}\psi^{j+}\psi^{k-}\psi^{l+} - \frac{\partial^2 W}{\partial\phi^k\partial\phi^l}\psi^{k-}\psi^{l+}\,.
\end{aligned}
\tag{A.3}
$$

The EOM of the auxiliary field D^k from this lagrangian is algebraically solved in terms of ϕ^i, ψ^i as follows

$$
D^k = -\Gamma^k_{ij}\psi^{i-}\psi^{j+} - G^{kl}\frac{\partial W}{\partial\phi^l}\,, \tag{A.4}
$$

where $\Gamma^k_{ij}(\phi)$ denotes the Christoffel symbols of the metric $G_{ij}(\phi)$. Elimination of D^k using the above equation in the last lagrangian then gives the effective lagrangian of the dynamical variables ϕ^i, ψ^i

$$
\begin{aligned}
\mathcal{L}[\phi,\psi] = & \frac{1}{2}G_{ij}(\phi)\left(\partial_+\phi^i\partial_-\phi^j - i\psi^{i+}\partial_+\psi^{j+} - i\psi^{i-}\partial_-\psi^{j-}\right) + \\
& +\frac{1}{2}\left(\Gamma^k_{ij}\psi^{i-}\psi^{j+} + G^{kl}\frac{\partial W}{\partial\phi^l}\right)^2 - \\
& -\frac{i}{2}\frac{\partial G_{ij}}{\partial\phi^k}\left(\partial_+\phi^i\psi^{j+}\psi^{k+} + \partial_-\phi^i\psi^{j-}\psi^{k-}\right) + \\
& +\frac{1}{2}\frac{\partial^2 G_{ij}}{\partial\phi^k\partial\phi^l}\psi^{i-}\psi^{j+}\psi^{k-}\psi^{l+} - \frac{\partial^2 W}{\partial\phi^k\partial\phi^l}\psi^{k-}\psi^{l+}\,,
\end{aligned}
\tag{A.5}
$$

where in the second line the square is defined using the metric G_{ij}. This lagrangian is invariant (up to total derivative terms) under the on-shell rigid SUSY transformations

$$
\delta_\epsilon\phi^i = \bar{\epsilon}\psi^i\,, \qquad \delta_\epsilon\psi^i = -\left(\gamma^\mu\partial_\mu\phi^i + G^{ij}\frac{\partial W}{\partial\phi^j} + \Gamma^i_{jk}\psi^{j-}\psi^{k+}\right)\epsilon\,. \tag{A.6}
$$

The corresponding Noether supercharges for these transformations are found to be

$$
Q_\pm = \int dx\left(G_{ij}(\phi)\partial_\pm\phi^i\psi^{j\pm} \pm i\frac{\partial W(\phi)}{\partial\phi^i}\psi^{i\mp}\right). \tag{A.7}
$$

The SUSY configurations are defined as those satisfying the conditions

$$
\delta_\epsilon\phi^i = 0\,, \qquad \delta_\epsilon\psi^i = 0\,, \tag{A.8}
$$

for some ϵ. The first condition is trivially satisfied for $\psi^i = 0$ which is also allowed by the EOM of fermions. The second condition then restricts the SUSY parameter ϵ and leads to the BPS equations (see below). In the language of superfields the SUSY preservation condition can be restated as follows

$$
(\bar{\epsilon}\mathcal{Q})R^i = 0, \tag{A.9}
$$

where $\mathcal{Q}_\pm$ are the ordinary SUSY generators given by the eq. (2.2).

The purely bosonic lagrangian of the scalar fields is obtained by putting $\psi^i = 0$ in eq. (A.5) which gives

$$\mathcal{L}_B[\phi] = \frac{1}{2} G_{ij}(\phi) \partial_+ \phi^i \partial_- \phi^j + V(\phi) . \tag{A.10}$$

Here the effective potential of the scalar fields $V(\phi)$ is defined as

$$V(\phi) \equiv \frac{1}{2} G^{ij}(\phi) \frac{\partial W(\phi)}{\partial \phi^i} \frac{\partial W(\phi)}{\partial \phi^j} . \tag{A.11}$$

For a target metric $G_{ij}(\phi)$ of the euclidean signature and with a smooth analytic superpotential $W(\phi)$, the scalar potential $V(\phi)$ is always positive definite ($V(\phi) \geq 0$) and it vanishes *only* at the extrema of $W(\phi)$; i.e.

$$V(\phi) = 0 \qquad \Leftrightarrow \qquad \frac{\partial W(\phi)}{\partial \phi^i} = 0 . \tag{A.12}$$

The extrema of $W(\phi)$ thus correspond to the totally supersymmetric vacua of the original theory defined by the eq. (A.2). For a target space metric with lorentzian signature, these extrema still define the fully supersymmetric solutions to the EOM, though they are not (necessarily) describing the vacuum states.

The BPS states are defined as the field configurations minimizing the energy functional of the system. With a suitable choice of the reference frame, these are written as time independent configurations whose energy density using eqs. (A.10), (A.11) can be written as[8]

$$\mathcal{H}_B = \frac{1}{2} G^{ij} \left(G_{ik} \frac{d\phi^k}{dx} \mp \frac{\partial W}{\partial \phi^i} \right) \left(G_{jl} \frac{d\phi^l}{dx} \mp \frac{\partial W}{\partial \phi^j} \right) \pm \frac{\partial W}{\partial \phi^k} \frac{d\phi^k}{dx} . \tag{A.13}$$

This leads to a mass functional of the form

$$\mathcal{M} = \frac{1}{2} \int dx \left| G_{ik} \frac{d\phi^k}{dx} \mp \frac{\partial W}{\partial \phi^i} \right|^2 \pm [W(\phi_\infty) - W(\phi_{-\infty})] , \tag{A.14}$$

where the norm is defined using the metric G_{ij} and $\phi_{\pm\infty}$ denote the boundary values of $\phi(x)$ at $x \to \pm\infty$. Obviously, for a G_{ij} of euclidean signature, the absolute minima of the mass functional are given by field configurations on which the first term (with one of the two signs appropriate to a positive $\mathcal{M}$) vanishes, i.e.

$$G_{ik} \frac{d\phi^k}{dx} = \pm \frac{\partial W}{\partial \phi^i} . \tag{A.15}$$

This introduces a set of first order differential equations which are known as BPS equations. We could derive this equation also by algebraic arguments. By interpreting $Q_\pm$ of eq. (A.7) as the operators on a quantum Hilbert space, and using similar expressions for the conserved

[8] Here we treat x as a spatial coordinate while taking y as a euclidean time coordinate. Strictly speaking, the definition of BPS states in terms of the minima of energy is valid only in the case of a Minkowski space which has a definite time coordinate. Nevertheless, we can continue to define BPS states, even in a euclidean space, as 1/2 SUSY configurations and this will give the same BPS equations.

charges of translation $P_\pm$, it is easy to see that on a general field configuration they satisfy the following algebra

$$\{Q_+, Q_-\} = \mathcal{Z}\,, \qquad Q_\pm^2 = -P_\pm\,, \tag{A.16}$$

where the central charge $\mathcal{Z}$ of the algebra is given in terms of the boundary values of ϕ^i as follows

$$\mathcal{Z} = 2\,[W(\phi_\infty) - W(\phi_{-\infty})]\,. \tag{A.17}$$

Apart from its sign, this is just twice the last term in eq. (A.14). Using the above algebra it is then easy to show that the mass squared operator $\mathcal{M}^2 \equiv P_+P_-$ takes the following form

$$\mathcal{M}^2 = \frac{1}{4}[(\bar{Q}Q)^2 + \mathcal{Z}^2]\,. \tag{A.18}$$

(Here the definition of $\bar{Q}Q$ needs an additional i factor appropriate to a Minkowski space as compared to a euclidean space.) According to this formula, mass of all states satisfies a lower bound, $\mathcal{M} \geq \frac{1}{2}|\mathcal{Z}|$, in agreement with the bound predicted from the eq. (A.14). The BPS bound $\mathcal{M} = \frac{1}{2}|\mathcal{Z}|$ is saturated just on the states on which $Q_+ = \mathrm{e}^{i\gamma}Q_-$, with γ being an arbitrary phase which can be set to $\pm i$ by suitable rotations. The BPS condition then becomes

$$(Q_+ \pm iQ_-)|BPS\rangle = 0\,. \tag{A.19}$$

Using the expressions (A.7) for $Q_\pm$, we can then verify that a single particle BPS state classically satisfies the BPS equation (eq. (A.15)). The same equation could be alternatively derived from the SUSY preservation condition, eqs. (A.8). The $\delta_\epsilon\phi^i = 0$ condition is trivially satisfied by putting $\psi^i = 0$ while the $\delta_\epsilon\psi^i = 0$ conditions requires

$$\begin{aligned} i\epsilon^+\partial_+\phi^i + \epsilon^- G^{ik}\frac{\partial W}{\partial\phi^k} &= 0\,, \\ -i\epsilon^-\partial_-\phi^i + \epsilon^+ G^{ik}\frac{\partial W}{\partial\phi^k} &= 0\,. \end{aligned} \tag{A.20}$$

This once again gives the BPS equation together with the subsidiary condition $\epsilon^+ = \pm i\epsilon^-$.The latter condition means that BPS states of the σ-model (A.2) preserve only 1/2 of the total SUSY that underlies this theory. Indeed, by using this relation between ϵ^+, ϵ^-, the SUSY preservation condition in the form of eq. (A.9) takes a form similar to eq. (A.19), namely $(\mathcal{Q}_+ \pm i\mathcal{Q}_-)R^i = 0$.

We now give a proof that the BPS states defined by eq. (A.15) actually also satisfy the classical EOM. For one dimensional solutions depending only on $x \equiv x^+ + x^-$ the EOM of the bosonic theory (A.10) is written in the form of a geodesic equation with an external force derived from V,

$$G_{ij}\frac{d^2\phi^j}{dx^2} + \Gamma_{i|jk}\frac{d\phi^i}{dx}\frac{d\phi^j}{dx} = \frac{\partial V}{\partial\phi^i}\,. \tag{A.21}$$

On the other hand, the derivative of the BPS equation (say with $+$ sign) with respect to x after simple manipulations becomes

$$G_{ij}\frac{d^2\phi^j}{dx^2} + \Gamma_{i|jk}\frac{d\phi^i}{dx}\frac{d\phi^j}{dx} = \frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\frac{d\phi^j}{dx} - \frac{1}{2}\frac{\partial G_{jk}}{\partial\phi^i}\frac{d\phi^j}{dx}\frac{d\phi^k}{dx}\,. \tag{A.22}$$

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The l.h.s. of this equation is just the l.h.s. of the EOM. The r.h.s. now can be written using the same BPS equation as

$$\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}G^{jk}\frac{\partial W}{\partial\phi^k} - \frac{1}{2}\frac{\partial G_{jk}}{\partial\phi^i}G^{jl}G^{km}\frac{\partial W}{\partial\phi^l}\frac{\partial W}{\partial\phi^m} = \frac{\partial}{\partial\phi^i}\left(\frac{1}{2}G^{jk}\frac{\partial W}{\partial\phi^j}\frac{\partial W}{\partial\phi^k}\right) = \frac{\partial V}{\partial\phi^i}\,, \tag{A.23}$$

which is nothing but the r.h.s. of the EOM, hence proving our assertion.

Finally, we give the proofs of the two facts used in the text; namely that in terms of the superfields R^i, the BPS equation is written just in the same form as eq. (A.15) for the scalar components ϕ^i,

$$G_{ik}(R)\frac{dR^k}{dx} = \pm\frac{\partial W(R)}{\partial R^i}\,, \tag{A.24}$$

and further that each pair of R^i's satisfy the following condition

$$\mathcal{D}_+R^i\mathcal{D}_-R^j = 0\,. \tag{A.25}$$

The proofs of both facts easily follow from the simple expression that R^i's take on a BPS configuration, namely

$$R^i = \phi^i - \theta^+\theta^- W^i(\phi)\,, \tag{A.26}$$

where $W^i(\phi) \equiv G^{ik}(\phi)\partial W/\partial\phi^k$ and $\phi^i(x)$ satisfies the BPS equation in the form $d\phi^i/dx = \pm W^i(\phi)$. Taking the derivative of this R^i with respect to x using the BPS equation then gives

$$\frac{dR^i}{dx} = \frac{d\phi^i}{dx} - \theta^+\theta^-\frac{d\phi^k}{dx}\frac{\partial W^i(\phi)}{\partial\phi^k} = \pm\left(W^i(\phi) - \theta^+\theta^- W^k(\phi)\frac{\partial W^i(\phi)}{\partial\phi^k}\right). \tag{A.27}$$

The parenthesis on the r.h.s. is obviously the θ-expansion of $W^i(R)$ and so we find eq. (A.24) with the indices raised with $G^{ij}(R)$. We now compute the supercovariant derivatives of R^i using eq. (3.2) on the same expression as in eq. (A.26). This gives

$$\begin{aligned}\mathcal{D}_+R^i &= -\theta^- W^i(\phi) - i\theta^+\frac{d\phi^i}{dx} = W^i(\phi)(-\theta^- \mp i\theta^+)\,,\\ \mathcal{D}_-R^i &= \theta^+ W^i(\phi) - i\theta^-\frac{d\phi^i}{dx} = W^i(\phi)(\theta^+ \mp i\theta^-)\,.\end{aligned} \tag{A.28}$$

Due to the proportionality of all the derivatives to a grassmann odd quantity, the products of each pair of them is vanishing and so eq. (A.25) follows.

References

[1] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[2] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[3] V.P. Nair and A.P. Polychronakos, *Quantum mechanics on the noncommutative plane and sphere*, *Phys. Lett.* **B 505** (2001) 267 [hep-th/0011172].

[4] G.V. Dunne, R. Jackiw and C.A. Trugenberger, *'Topological' (Chern-Simons) quantum mechanics*, *Phys. Rev.* **D 41** (1990) 661.

[5] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, *Phys. Rev.* **D 62** (2000) 066004 [hep-th/9908056].

[6] N. Seiberg, *A note on background independence in noncommutative gauge theories, matrix model and tachyon condensation*, *J. High Energy Phys.* **09** (2000) 003 [hep-th/0008013].

[7] C.-S. Chu and F. Zamora, *Manifest supersymmetry in non-commutative geometry*, *J. High Energy Phys.* **02** (2000) 022 [hep-th/9912153].

[8] S. Ferrara and M.A. Lledó, *Some aspects of deformations of supersymmetric field theories*, *J. High Energy Phys.* **05** (2000) 008 [hep-th/0002084].

[9] M.A. Lledó, *Supersymmetric field theories on deformed space-time*, hep-th/0011268.

[10] J. Bagger and I. Giannakis, *Spacetime supersymmetry in a nontrivial NS-NS superstring background*, *Phys. Rev.* **D 65** (2002) 046002 [hep-th/0107260].

[11] D. Klemm, S. Penati and L. Tamassia, *Non(anti)commutative superspace*, hep-th/0104190.

[12] P. Kosinski, J. Lukierski and P. Maslanka, *Quantum deformations of space-time SUSY and noncommutative superfield theory*, hep-th/0011053.

[13] R. Abbaspur, *Noncommutative supersymmetry in two dimensions*, *Int. J. Mod. Phys.* **A 18** (2003) 855 [hep-th/0110005].

[14] R. Abbaspur, *Generalized noncommutative superalgebras*, *Mod. Phys. Lett.* **A 18** (2003) 587; *A field theoretic realization of the generalized noncommutative supersymmetry*, to appear.

[15] R. Abbaspur, in preparation.

[16] S. Paban, S. Sethi and M. Stern, *Non-commutativity and supersymmetry*, *J. High Energy Phys.* **03** (2002) 012 [hep-th/0201259].

[17] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, hep-th/0302078.

[18] S.J. Gates Jr., et al., *Superspace: one thousand and one lessons in supersymmetry*, Benjamin/Cummings, London 1983.

[19] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, 1983 (second edition 1992).

[20] E. Witten and D.I. Olive, *Supersymmetry algebras that include topological charges*, *Phys. Lett.* **B 78** (1978) 97.

[21] B. Chibisov and M.A. Shifman, *BPS-saturated walls in supersymmetric theories*, *Phys. Rev.* **D 56** (1997) 7990 [hep-th/9706141], erratum *ibid.* **D58** (1998) 109901 [hep-th/9706141].

[22] M. Naganuma, M. Nitta and N. Sakai, *BPS walls and junctions in SUSY nonlinear sigma models*, *Phys. Rev.* **D 65** (2002) 045016 [hep-th/0108179].

[23] R. Rajaraman, *Solitons and instantons: an introduction to solitons and instantons in quantum field theory*, North-Holland, Amsterdam 1989.

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